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AN INTRINSIC FORMULA FOR VOLUME

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1. Introduction and summary. Previous algorithms for the computation of the volume of a specified domain in E^n involve extraneous geometrical elements which are not intrinsically connected with the domain. That this must be so if the volume is to be determined, fundamentally, by a partitioning of the domain is sufficiently clear, for each such partitioning involves the specification of arbitrary extraneous elements such as an n-uple system of surfaces (corresponding to the cellular division of the domain associated with the "curvilinear coordinates" imposed on E^n by the surfaces), an (n-1)-flat and distances from that flat (corresponding to a partitioning into elementary right cylinders with generating lines orthogonal to the flat), poles and angles if the volume is determined in terms of the solid angles subtended at a specified pole by elements of the boundary of the domain (this corresponds to a partitioning into elementary cones, or portions of cones, with vertex at the pole), etc. A radically different type of volume determination, not involving partitioning, is provided by Crofton's beautiful formula (1869) for the area of a convex figure in the plane and the analogous formula for the volume of a convex body in E^3 due to Varga (1936). However, these formulae are still far from being intrinsic, involving as they do the specification of the length of chord intercepted by the domain for each line in the plane or 3-space in terms of the 'coordinates' of that line (for n=2, distance of the line from an arbitrary pole and its orientation relative to an arbitrary axis, and for n=3, the trace of the line on an arbitrary plane and the angles between the line and two arbitrary orthogonal axes in that plane).

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[†]The main result in this note is a byproduct of some new results in geometrical probability which I hope to publish shortly.

On the other hand, the volume of a given domain is an intrinsic feature of the domain which does not depend on foreign elements extraneous to it. In particular, the volume is invariant under rigid motions (congruent transformations, or translations and rotations) and reflections, and any intrinsic computational formula for volume, to justify the name, should reflect this fundamental feature in its very structure (and not merely in the general theory of volume). This is not the case with any non-intrinsic formula (such as those mentioned in the previous paragraph), since it is not then obvious that the formula will continue to produce the same number under translations, rotations and reflections.

What, then, should an "intrinsic" formula for volume (assuming one exists) look like? Roughly speaking, the volume of a bounded domain is the amount of space enclosed by the boundary, and is moreover determined entirely by the nature of the boundary. This enables the notion of intrinsic formula for volume to be made more precise: A formula for volume is intrinsic if it involves only intrinsic properties of the boundary, that is, properties which pertain to the boundary only and not to the surrounding space in which the boundary is imbedded. Such a formula might perhaps incorporate global or local properties of the boundary, for example, curvature, which are invariant under the class of transformations mentioned previously, and might also incorporate symmetric properties between pairs of boundary points which are similarly invariant. Two such symmetric invariant properties are the distance, ρ, between two points on the boundary and the angle, ψ, between the tangent flats (if these exist), suitably oriented, at the points. In this note, I obtain an intrinsic formula for volume, under some weak restrictions on the nature of the domain, in terms of ρ and ψ only. (Rather remarkably, no other geometrical properties of the boundary enter into the formula.) A slightly more general, though very similar, formula is also obtained for the product of the volumes of two (not necessarily disjoint) domains. To the best of my knowledge, no intrinsic formulae (in the present sense) for volume have appeared previously.

It may be helpful to summarize here the main argument of this note. We show first that each domain R in E^n belonging to a certain class $\mathfrak{R}^n$ defines uniquely a function $\underset{\sim}{g}_R$ on E^n, with range E^n, in terms of a vector integral on the set of points P on the boundary of R at which a normal exists, such that the divergence of $\underset{\sim}{g}_R$ is everywhere constant and in fact equal to the volume of R. ($\underset{\sim}{g}_R$ is defined in Equ. (4).) Consequently, the integral of the divergence of $\underset{\sim}{g}_R$ over R gives the <u>square</u> of the volume of R. The latter integral is transformed by means of the Gauss divergence theorem in n-dimensional form into an iterated integral on P and P (or a 'multiple' integral on $P \times P$).

<u>Notation and terminology</u>.

(i) The term <u>standard domain</u> shall be understood in the sense of Whitney (1957); that is, R is a standard domain in E^n if it is a bounded connected open set and† $\partial R = P \cup Q$, where $Q \subset \bar{R}-R$ is a closed set of zero (n-1)-extent and $P = (\bar{R}-R)-Q$ is the union of a finite or countable set of smooth manifolds with the property that at each point of each manifold there exists a neighbourhood such that the set of points of R in this neighbourhood lies just to one side of the manifold. (For a definition of zero s-extent, see Whitney, p. 97. In most applications, Q will consist of a finite set of manifolds, each with dimension $< n-1$, and will then automatically have zero (n-1)-extent.) We note in particular that the outward normal is well defined at every point in P.

We write $R \in \mathfrak{R}^n$ if R is a standard domain in E^n and P has finite (n-1)-volume.

(ii) We use the dot notation for inner product (scalar product) of vectors in E^n, i.e., if $\underset{\sim}{a},\underset{\sim}{b}$ are such vectors then $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

(iii) ∇ and $\nabla\cdot$ denote the gradient and divergence operators in E^n; that is, if $\underset{\sim}{r} = (x_1,\ldots,x_n)\in E^n$, f is a real-valued function on E^n and $g = (g_1,\ldots,g_n)$ is

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† As usual, $\bar{R}$ and $\partial R = \bar{R} \cap \overline{E^n - R}$ denote the closure and boundary of R.

a function on E^n with range in E^n, then

$$\nabla f = \sum_{\alpha=1}^{n} \frac{\partial f}{\partial x_\alpha} \underset{\sim}{e}_\alpha , \qquad \nabla \cdot \underset{\sim}{g} = \sum_{\alpha=1}^{n} \frac{\partial g_\alpha}{\partial x_\alpha} ,$$

where $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_n$ are unit vectors parallel to the (orthogonal) coordinate axes.

If $f = f(\underset{\sim}{r}, \underset{\sim}{r}')$ is a real-valued function on $E^n \times E^n (\underset{\sim}{r} \epsilon E^n, \underset{\sim}{r}' \epsilon E^n)$, $\underset{\sim}{g} = \underset{\sim}{g}(\underset{\sim}{r}, \underset{\sim}{r}')$ a function on $E^n \times E^n$ with values in E^n, then ∇f shall denote the gradient of f with $\underset{\sim}{r}$ a variable point and the point $\underset{\sim}{r}'$ a fixed pole, $\nabla' f$ the gradient of f with $\underset{\sim}{r}'$ a variable point and the point $\underset{\sim}{r}$ a fixed pole, and similar meanings shall attach to $\nabla \cdot \underset{\sim}{g}$ and $\nabla' \cdot \underset{\sim}{g}$.

2. Statement and proof of basic result.

THEOREM. If $R \in \mathfrak{R}^n$ $(n \geq 1)$, then its volume, $V(R)$, is given by

$$V^2(R) = -\frac{1}{2n} \int_P \int_P |\underset{\sim}{r} - \underset{\sim}{r}'|^2 \cos \psi d\sigma d\sigma' , \tag{A}$$

where $d\sigma$, $d\sigma'$ are the $(n-1)$-volumes of elements at the points $\underset{\sim}{r}$, $\underset{\sim}{r}'$ in P and ψ is the angle between the outward normals at these points.

Proof. Since

$$\nabla \cdot \left(\frac{\underset{\sim}{r} - \underset{\sim}{r}'}{n}\right) = 1 ,$$

we have

$$V(R) = \int_R \nabla \cdot \left(\frac{\underset{\sim}{r} - \underset{\sim}{r}'}{n}\right) dv ,$$

where dv is the n-volume of an element at the point $\underset{\sim}{r}$ in R. Hence, from the Gauss divergence theorem in n-dimensional form† (Whitney, 1957; pp. 100-101),

$$V(R) = \int_P \left(\frac{\underset{\sim}{r} - \underset{\sim}{r}'}{n}\right) \cdot \underset{\sim}{w} d\sigma , \tag{1}$$

$\underset{\sim}{w}$ denoting the unit outward normal vector at the point $\underset{\sim}{r}$, with $\underset{\sim}{r}'$ fixed. But

$$\frac{\underset{\sim}{r} - \underset{\sim}{r}'}{n} = \nabla'\left(-\frac{|\underset{\sim}{r} - \underset{\sim}{r}'|^2}{2n}\right) .$$

Therefore

$$V(R) = \int_P \nabla'\left(-\frac{|\underset{\sim}{r} - \underset{\sim}{r}'|^2}{2n}\right) \cdot \underset{\sim}{w} d\sigma . \tag{2}$$

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† (1) is otherwise obvious and the divergence theorem need not be used at this point in the proof. In fact (1) simply says that $V = \frac{1}{n} \int_P |\underset{\sim}{r} - \underset{\sim}{r}'| \cos\theta \, d\sigma$, where θ is the angle between $\underset{\sim}{r} - \underset{\sim}{r}'$ and the outward normal at the point $\underset{\sim}{r}$ on P, a result which is by no means unknown (to say the least!) and which follows by partitioning R into elementary cones, or portions of such cones, with common vetex at the point $\underset{\sim}{r}'$. We use the divergence theorem for the sake of consistency.

Assume that $|\underset{\sim}{r}'|$ is bounded. Then the boundedness of $|\underset{\sim}{r} - \underset{\sim}{r}'|$ for $\underset{\sim}{r} \in R$ and all admissible $\underset{\sim}{r}'$, together with the boundedness of $\int_P d\sigma$, gives through (2) ($\underset{\sim}{w}$ being independent of $\underset{\sim}{r}'$),

$$V(R) = \nabla' \cdot \int_P \left(- \frac{|\underset{\sim}{r} - \underset{\sim}{r}'|^2}{2n} \right) \underset{\sim}{w} d\sigma \; ; \tag{3}$$

that is, the function $\underset{\sim}{g}_R(\underset{\sim}{r}')$, defined on E^n by

$$\underset{\sim}{g}_R(\underset{\sim}{r}') = \int_P \left(- \frac{|\underset{\sim}{r} - \underset{\sim}{r}'|^2}{2n} \right) \underset{\sim}{w} d\sigma \; , \tag{4}$$

satisfies the equation

$$\operatorname{div} \underset{\sim}{g}_R(\underset{\sim}{r}') \equiv \nabla' \cdot \underset{\sim}{g}_R(\underset{\sim}{r}') = V(R) \; . \tag{5}$$

Thus, using (5) and the Gauss divergence theorem again,

$$\begin{aligned} V^2(R) &= \int_R [\nabla' \cdot \underset{\sim}{g}_R(\underset{\sim}{r}')] dv' \\ &= \int_P \underset{\sim}{g}_R(\underset{\sim}{r}') \cdot \underset{\sim}{w}' d\sigma' \; , \end{aligned} \tag{6}$$

where dv' is the n-volume of an element at a point $\underset{\sim}{r}' \in R$ and $\underset{\sim}{w}'$ is the unit outward normal at a point $\underset{\sim}{r}' \in P$. Substitution of $\underset{\sim}{g}_R(\underset{\sim}{r}')$ from (4) in (6) then gives

$$V^2(R) = \int_P \int_P \left(- \frac{|\underset{\sim}{r} - \underset{\sim}{r}'|^2}{2n} \right) \underset{\sim}{w} \cdot \underset{\sim}{w}' d\sigma d\sigma' \; .$$

This completes the proof.

3. Further remarks.

(1) If $\nabla' \cdot g_R(r')$ is integrated over any domain $R' \in \mathfrak{R}^n$ (rather than, as in the proof of the Theorem, over R), then we obtain the slightly more general result

$$V(R)V(R') = -\frac{1}{2n} \int_{r' \in P'} \int_{r \in P} |r - r'|^2 \cos\psi \, d\sigma d\sigma' , \qquad (B)$$

$d\sigma'$ denoting here the $(n-1)$-volume of an element at the point r' in P' (P' is related to R' in the same way that P is to R.) (B) reduces to (A) when $R' = R$.

However, it should be noted that since separate rigid transformations of R and R' will not, in general, preserve $|r - r'|$ and ψ, it is not evident that the right member of (B) is invariant under such transformations (though, of course, this must be the case, the latter quantity being equal to $V(R)V(R')$). This is to be contrasted to formula (A), relating to a single domain, in which the invariantive property under rigid transformations of the right member of (A) is manifest, $|r - r'|$ and ψ being here clearly preserved under such transformations.

(2) Formulae (A) and (B) can be obtained under even weaker restrictions on R and R' by using a more general variant of the Gauss divergence theorem in E^n (See Federer, 1945). I have refrained from doing so for reasons of simplicity.

(3) Formulae (A) and (B) have been obtained as iterated integrals. In view of the restrictions on R and R', it is evident that they can also be viewed, more symmetrically, as "multiple" integrals over $P \times P$ and $P \times P'$, respectively.

(4) (A) and (B) can be expressed in terms of the mathematical expectation of $|r - r'|^2 \cos\psi$, where r and r' are here independent stochastic variables with uniform densities on P and P in (A), and

on P and P' in (B); in other words, $(\underset{\sim}{r},\underset{\sim}{r}')$ is assigned a uniform density on $P \times P$ or on $P \times P'$ (ψ is a measurable function of $(\underset{\sim}{r},\underset{\sim}{r}')$). The relevant formulae are

$$E[|\underset{\sim}{r} - \underset{\sim}{r}'|^2 \cos \psi] = (-2n) \frac{V^2(R)}{S^2(R)} \qquad (A')$$

and

$$E[|\underset{\sim}{r} - \underset{\sim}{r}'|^2 \cos \psi] = (-2n) \frac{V(R)}{S(R)} \frac{V(R')}{S(R')} , \qquad (B')$$

$S(R)$, $S(R')$ denoting the $(n-1)$-volumes of P and P'.

(5) We note from (A) and (B) that mutually orthogonal elements of the boundary (or boundaries) do not contribute to volume.

(6) In (A), let R be a polytope in E^n with k faces $F_1,\ldots,F_k$. Then the angle between every two elements on F_i and F_j is θ_{ij}, where θ_{ij} is the angle between the outward normals to F_i and $F_j (\theta_{ii} = 0)$. Thus (A) reduces to

$$-2n\ V^2(R) = \sum_1^k C_{ii} + 2 \sum_{j>i}^k C_{ij} \cos \theta_{ij} , \qquad (7)$$

where

$$C_{ij} = \int_{F_i} \int_{F_j} |\underset{\sim}{r}_i - \underset{\sim}{r}_j|^2\, d\sigma_j d\sigma_i \qquad (i,j = 1,\ldots,k),$$

$\underset{\sim}{r}_i$ denoting a point on F_i and $d\sigma_i$ the $(n-1)$-volume of an element at $\underset{\sim}{r}_i$ in F_i. Similarly, if R and R' are two polytopes in E^n with faces $F_1,\ldots,F_k$ and $F'_1,\ldots,F'_{k'}$, (B) reduces to

$$-2n\ V(R)V(R') = \sum_{i=1}^{k} \sum_{j=1}^{k'} C_{ij} \cos \theta_{ij} , \qquad (8)$$

where θ_{ij} is the angle between the outward normals of F_i and F'_j and

$$C_{ij} = \int_{F_i} \int_{F'_j} |\underset{\sim}{r}_i - \underset{\sim}{r}'_j|^2\, d\sigma'_j d\sigma_i \qquad (i = 1,\ldots,k;\ j = 1,\ldots,k').$$

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